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with the regards of
The Author

REFRACTION IN THE PRINCIPAL MERIDIANS OF A TRIAXIAL ELLIPSOID, WITH REMARKS ON THE CORRECTION OF ASTIGMATISM BY CYLINDRICAL GLASSES; AND AN HISTORICAL NOTE ON CORNEAL ASTIGMATISM.

By SWAN M. BURNETT, M.D., of Washington.

WITH A COMMUNICATION ON THE MONOCHROMATIC ABERRATION OF THE HUMAN EYE IN APHAKIA.

By Prof. WM. HARKNESS,

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IN considering refraction in the principal meridians of an ellipsoid with three unequal axes, as we have it in regular corneal astigmatism, all the writers with whom I am familiar—including Helmholtz, Knapp, Donders, Aubert, Fick, and Mauthner—have tacitly assumed that the refraction in these two meridians was free from monochromatic aberration.

It seemed to me, however, that as these meridians have radii which, from the nature of the curved surface, change their length at each successive point, we must have a refraction differing in some respects from the spherical.

As the matter is one very easily and simply settled by construction, I have endeavored to determine the character of such aberration.

We have, of course, from a purely optical point of view, two cases to deal with: one in which the light falls on the sharper end of the ellipsoid, or in the direction of the long axis; and one in which the light falls on the flatter end, or in the direction of the short axis.

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The laws for determining the course of any ray of light, after its refraction by a curved surface, are very simple. We have only to know the index of refraction of the refracting medium, and the angle the incident ray makes with the normal to the surface at the point of incidence; the law of sines then gives us at once the direction of the refracted ray.

In the case of the circle, or a section of it, we find the normal by drawing a line from the centre, through the point of incidence, in other words, by prolonging the radius. In the case of an ellipse, however, we can not get the normal in so simple a manner, but there is a well-known theorem,[1] which enables us to do it quite readily. According to this theorem, *all circles and ellipses, whose diameters and major axes correspond, have the same subtangents.* We have constructed fig. 1, which represents the sharper end of the

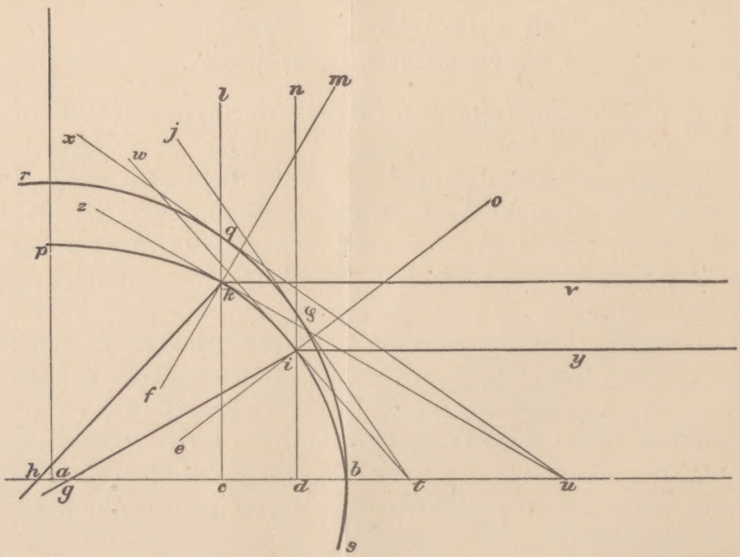

FIG. 1.

ellipse—that is, one of the principal meridians of the ellipsoid,

[1] To be found in any treatise on analytical geometry. Compare Loomis' "Elements of Analytical Geometry," 1873: page 113. "Since the subtangent is independent of the minor axis, it is the same for all ellipses which have the same major axis; and since the circle on the major axis may be considered as one of these ellipses, the subtangent is the same for an ellipse and its circumscribing circle."

—in accordance with this theorem. Let *ab* represent the major axis of the ellipse of which *bp* is a portion; from *a* as a centre and *ab* as a radius, draw the segment *rs* of a circle. Let *v* and *y* be the rays parallel to the axis and incident at *k* and *i*. Through the points *k* and *i*, draw *lc* and *nd* perpendicular to *ab*. These will cut the circle at *q* and *φ*, and the normals at these points coincide with lines drawn through them and the centre *a*; and the lines *xu* and *jt* drawn at right angles to these normals, will be the tangents at the points *q*, *φ*. Now, if the circle and ellipse have the same subtangents *bu* and *bt*, then the lines *zu* and *wt*, drawn through *u* and *k* and *t* and *i* must be the tangents to the ellipse at the points *k* and *i*, and the lines *mf* and *oe*, drawn perpendicular to them, must be normals to the surface at the points of incidence, *k* and *i*. We have now all the requisite data, and have only to apply the law of sines in order to find the course of the rays *ig* and *kh*. In this case, in order to have the diagram fall within reasonable limits, we have assumed a refractive index = 3.

It will be seen at a glance that in the case where the rays fall parallel to the long axis of the ellipse, the ray *iy*, nearer the axis, crosses the principal axis, *au*, after refraction, *in front* of the more peripheral ray *kv*,—that is to say, *we have an aberration the opposite in kind to that of an ordinary spherical surface.*

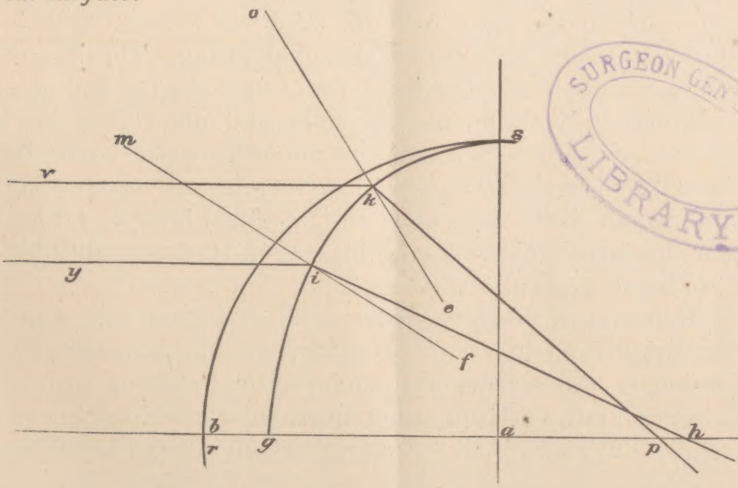

FIG. 2.

If, however, the light falls on the ellipse in the direction of the short axis, or on the blunter end, as we have it represented in fig. 2 (which has been constructed according to the same plan as fig. 1), we find that the more peripherally refracted ray *kp* crosses the principal axis in front of the more centrally refracted ray *ih;* in other words, we have an *excess of the ordinary spherical aberration.*

It therefore becomes evident, *that if we take a series of curves passing over from the flatter to the sharper end of an ellipse, we will have in the refraction, first, an exaggeration of the spherical aberration—which will be greater in proportion to the difference in the length of the major and minor axes—diminishing until the curve becomes a circle, when there will be only the ordinary amount of spherical aberration; then, as the minor axis becomes shorter, this aberration will still further diminish until it becomes for any chosen rays practically zero. As the minor axis still further shortens, the aberration passes over to an opposite kind, and the more central rays cross the principal axis in front of the more peripheral, and this will increase* PARI PASSU *with the shortening of the minor axis.*

In applying these principles to the eye, some interesting facts in regard to the retinal images of astigmatics and the influence of cylindrical lenses on this anomaly are made manifest.

In the first place, it is evident that a deviation of the cornea from the spherical form need not necessarily be injurious to distinctness of the retinal image. On the contrary, should it assume the form of the sharper end of an ellipsoid of revolution and the major and minor axes bear a certain ratio to each other, the monochromatic aberration would be practically abolished, a circumstance which would add much to the sharpness of the retinal image. Of how far this is true of the actual cornea, Prof. Harkness will speak in his communication.

If, however, this ratio between the major and minor axes is varied from in either direction, a monochromatic aberration is at once manifest, and for the relief of such an aberration (we are supposing now that the ellipsoid is one of revolution, and all the meridians are alike) no ordinary

cylindrical or spherical glass can be of any benefit. It is possible that some cases of amblyopia which are benefited but little or not at all by either spherical or cylindrical glasses, and where the nervous apparatus seems intact, fall in this category. We look upon some forms of conical cornea as exaggerated examples of this condition.

Where the cornea, as it usually does, represents a triaxial ellipsoid, we will have a different set of conditions according to the character of the curvature; and the action of cylindrical lenses on the refraction in the principal meridians will not be uniform in all cases.

Let us take, as an example, that form in which the cornea represents the sharper end of an ellipsoid with three unequal axes. It is plain from what has gone before, that the meridian of greater curvature, after it has passed a certain point, will suffer from the greater aberration. *A* in fig. 3 represents the meridian of less, and *B* the meridian of greater curvature. In *A*, the peripheral ray *d* crosses the principal

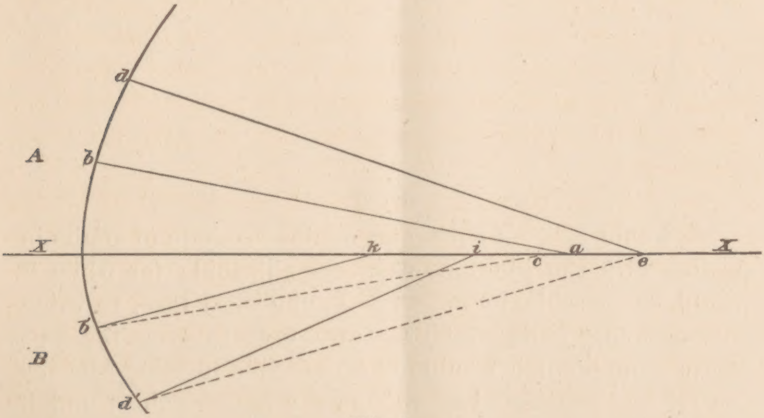

FIG. 3.

axis xx′ at *e* and the more central ray *b* at *a*, while in *B* the ray *d′* crosses at *i*, and *b′* at *k*. If now we place a cylindrical lens before the refracting surface with its curvature corresponding to the meridian *B*, and of such strength that the peripheral ray *d′* is carried back and made to cross the axis in the same point *e* as the peripheral ray *d* of the meridian *A*, the relation between *k* and *i*, though they are carried

back from their orignal position, remains unaltered at *c* and *e*, since the regular refraction of the cylinder does not counteract the aberration of the elliptical surface. The result would be that the rays crossing at *a* and *c*, would form figures of diffusion on the focal plane passing through *e*.

If we bring the more peripheral rays, *d* and *d'*, of the two meridians, *A* and *B*, to cross at the same point *c*, moving them forward from *a*, *i*, as in fig. 4, we have the same condition; for the central rays *b*, *b'*, which cross the axis at *o* and *k*, would form figures of diffusion on the focal plane passing through *c*.

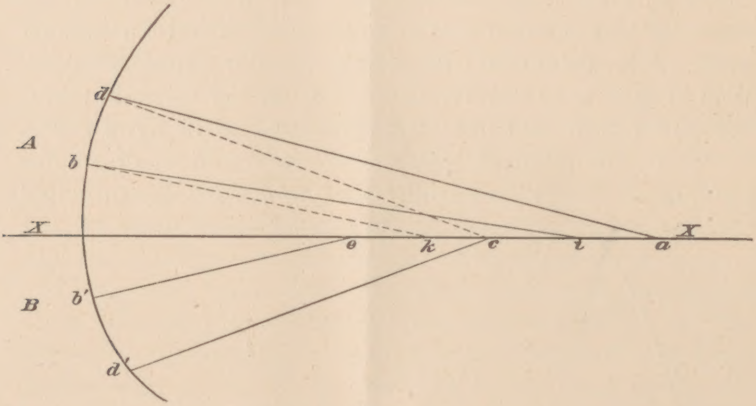

FIG. 4.

We would have, of course, an analogous state of affairs in dealing with the blunter end of the ellipsoid; for while it would be possible, by means of a cylindrical lens, to bring corresponding peripheral or central rays to cross the axis at the same point, it would not be possible to bring *both* the central and peripheral rays to cross it in one point; and if we should have to deal with a surface in which one meridian represented the blunter end of an ellipse, while the other represented the sharper end, the diffusion figures would be still more confusing. Fig. 5 represents such a surface where the peripheral rays, *d*, *d'*, are brought, by means of a cylinder, to cross the axis at the same point *e*.

The more central ray *b* of the flatter ellipse *A*, will cross the axis at *c*, *behind* the focal plane, passing through *e*, while

the more central ray b' of the sharper ellipse B, will cross it *in front* at a, thus forming two sets of diffusion figures.

Under any of these forms, which the cornea may assume,[1]

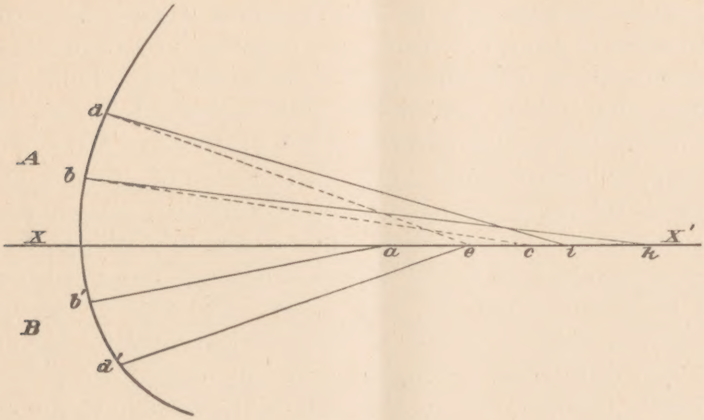

FIG. 5.

the retinal image must have its distinctness of outline impaired by the circles of diffusion which fall on it.[2] This diffusion being greater in the higher than in the lower forms of astigmatism, we should expect to find the visual acuteness, after all possible correction, less in the former, and such I believe is the experience of all practitioners. We should also expect that the vision of astigmatics, after correction, would be less than that of myopes and hypermetropes of the same grade after their neutralization by spherical lenses.

As a matter of statistics, I find that out of about 2,000 astigmatic eyes of all degrees, only about $\frac{1}{10}$ have $V = 1$, after the best possible correction.

It must also be apparent that this aberration in the principal meridians will be greater the greater the angular aperture—which in the eye would be represented by the pupil,—and consequently the larger the pupil the larger the

[1] In all the measurements that have been made up to the present time, the cornea has never been found to assume in either of its meridians the form of the blunter end of an ellipse ; but we see no reason to doubt the *possibility* of such an occurrence.

[2] We do not take into consideration here the rays passing through the intermediate meridians. These require a separate investigation.

figures of diffusion, and the more indistinct the image. It is for this reason, among others, that in examining for the presence, and particularly the degree of astigmatism, I always abstain, except on rare occasions, when I have the strongest reasons to suspect the presence of spasm of accommodation, from dilating the pupil. I know it has become the fashion, which has quite a large following, to paralyze the accommodation (and at the same time, of course, dilate the pupil) in the examination of almost every case of refractive anomaly, and especially where astigmatism is suspected. One principal argument against this is, that when the eye is under the influence of a mydriatic, it is not in its normal condition, and glasses fitted to such an eye may not, and as a matter of my experience usually do not, at all suit it when the effects of the mydriatic have passed away. To say nothing of the sources of error due to the enlarged pupil, there is a greater or less amount of tension of the ciliary muscle, normal to every eye, which disappears under the mydriatic, and gives the eye a refraction below that which it will have when it returns to its natural state.

Furthermore, I think, judging from my own experience, that spasm of accommodation is not common in astigmatism; because, except in the compound hypermetropic form, no benefit can accrue from such spasm.[1] It would be impossible for any amount of contraction of the ciliary muscle in its totality to overcome the difference in the refraction in the two meridians so as to give clear and distinct vision. If, however, a mydriatic is used, it is always wise to wait until the pupil has regained its normal size before the final glasses are ordered.

It will be remembered that there is one form of ellipse which is practically free from aberration, and it would be interesting to know how far the cornea of the emmetropic eye deviates from this form. The investigation of this question would lead into mathematical computations far beyond my skill. So I laid the subject before my friend, Dr. Wm.

[1] Except in the *partial* form spoken of by Dobrowolski and Javal; but of this we will speak at another time.

Harkness, Professor of Mathematics, U. S. N., to whose profound knowledge of optics I am already so much indebted. The results of his calculations are herewith subjoined.

ON THE MONOCHROMATIC ABERRATION OF THE HUMAN EYE IN APHAKIA.

A Letter to Dr. Swan M. Burnett by Wm. Harkness.

Some months ago you pointed out to me the want of exact knowledge among oculists respecting certain optical properties of the cornea of the human eye, and at your request I willingly undertook to investigate them. For that purpose you placed in my hands two works whose titles are as follow:

1. Vorlesungen über die optischen Fehler des Auges. Von Prof. Dr. Ludwig Mauthner, in Innsbruck. Wien, 1876.
2. On the Anomalies of Accommodation and Refraction of the Eye. By F. C. Donders, M.D., Sydenham Society, London, 1864.

For the sake of brevity these works will be designated respectively as Mauthner and Donders.

The cornea of the emmetropic eye seems to have an ellipsoidal form, but the existing data for determining its curvature in the vertical meridian are too meagre to give a satisfactory result. I have therefore confined my attention to the horizontal meridian, and have taken the data for it from table VII, upon pages 598–599 of Mauthner. That table exhibits the form and dimensions of the cornea in seventeen pairs of emmetropic eyes, and from the mean of these thirty-four eyes, it appears that in the visual axis the radius of curvature is 7.708 millimetres, while 20° to the inner side of that axis it is 8.378 millimetres, and 20° to the outer side 7.884 millimetres. Mauthner does not explain the phrase "20° to the inner (or outer) side of the visual axis," and I have had some trouble in ascertaining its true meaning.

It is customary to regard the outline of the cornea, along a horizontal section through the visual axis, as part of an ellipse. Let it be part of the ellipse NAMB, figure 6, of

which the major and minor semi-axes are respectively AH and BH; and let C, D, E, be the points at which the radii

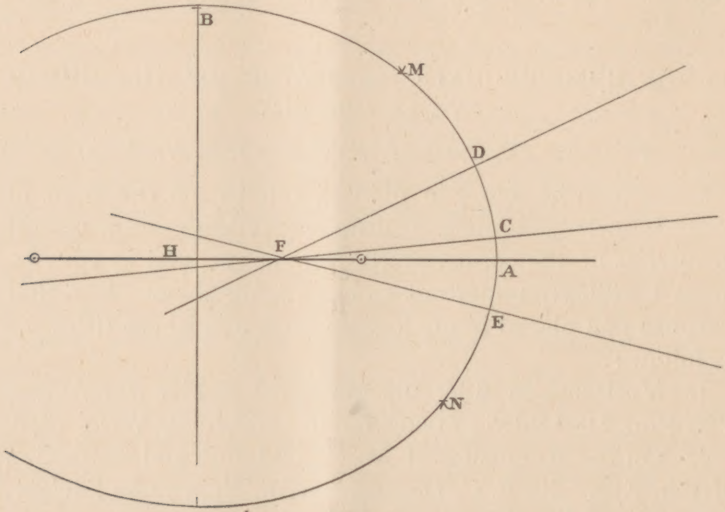

Fig. 6.

of curvature, r, r_1, r_2, are measured. DF, CF, and EF are normals to the ellipse at these points, and CF is the visual axis. By the phrase "20° to the inner side of the visual axis" Mauthner means that the angle CFD, between the visual axis and the normal at D, is 20°; and similarly, by "20° to the outer side of the visual axis" he means that the angle CFE is 20°. It may be well to remark that figure 6 is accurately drawn to scale, and the arc NAM represents that portion of the ellipse which is included within the limits of the cornea.

Let a represent the semi-major axis, AH; b the semi-minor axis, BH; φ the angle CFD = CFE; α the angle AFC, included between the major axis of the ellipse and the visual axis; r the radius of curvature in the visual axis; and r_1 and r_2 the radii of curvature at a distance of φ degrees on each side of the visual axis, r_1 being greater than r_2. Then, on pages 19 and 20 of an article entitled "Ueber die Accommodation des Auges," published in the *Archiv für Ophthalmologie*, Erster Band, Abtheilung 2, Helmholtz has shown that

$$\tan 2\alpha = \tan \varphi \frac{(rr_2)^{2/3} - (rr_1)^{2/3}}{(rr_1)^{2/3} + (rr_2)^{2/3} - 2(r_1 r_2)^{2/3}} \qquad (1)$$

$$\varepsilon^2 = \frac{r_1^{2/3} - r^{2/3}}{r_1^{2/3} \sin^2 (\alpha + \varphi) - r^{2/3} \sin^2 \alpha} \qquad (2)$$

$$a = \frac{r (1 - \varepsilon^2 \sin^2 \alpha)^{\frac{3}{2}}}{1 - \varepsilon^2} \qquad (3)$$

$$b = a (1 - \varepsilon^2)^{1/2} \qquad (4)$$

Substituting in these formulæ the radii of curvature given above, namely, $r = 7.708$ millimetres, $r_1 = 8.378$ millimetres, $r_2 = 7.884$ millimetres, we find for the normal cornea

$\alpha = 5^\circ \, 49' \, 52''$	$a = 10.9625$ millimetres.
$\varepsilon^2 = 0.300135$	$b = 9.1711$ "

We have next to consider how a system of parallel rays falling upon such a cornea, parallel to the visual axis, will be refracted; and as our object is to determine the errors of refraction produced by the cornea alone, we must conceive the lens to be absent. On account of the thinness of

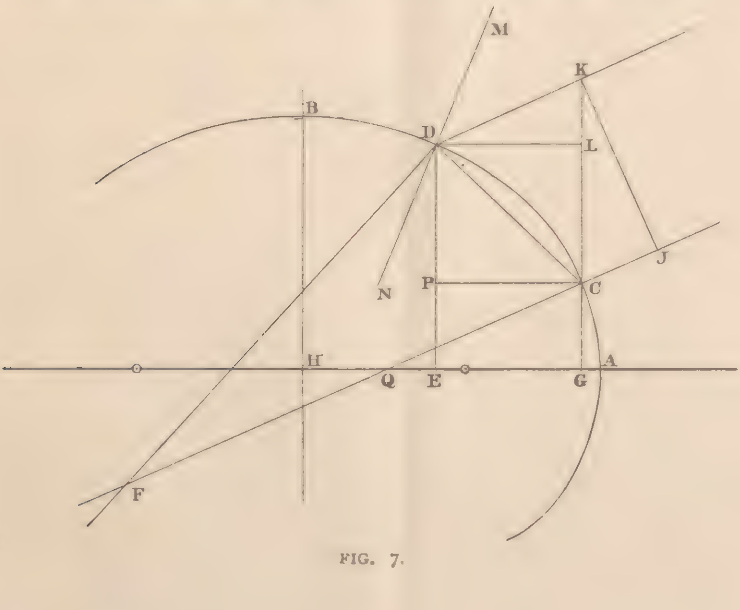

FIG. 7.

the cornea, the almost perfect parallelism of its surfaces, and the exceedingly slight difference in refractive power between it and the aqueous and vitreous humors, no sensible error will result if we regard the whole eye as a homogeneous solid, having the refractive index of the aqueous humor, and the same bounding surface as the cornea.

In fig. 7, let ACDB be one quarter of an ellipse, a portion of which forms the bounding surface of the cornea; AH being its semi-major, and BH its semi-minor, axis. Also let JCF be the visual axis, and KD a ray parallel to it, entering the cornea at the point D. The co-ordinates of the point C, at which the visual axis pierces the cornea, are HG and GC; and the co-ordinates of the point D are HE and ED. To find expressions for these co-ordinates, we assume the equation of an ellipse referred to its centre and axes, which is

$$0 = a^2y^2 + b^2x^2 - a^2b^2 \tag{5}$$

and in this we put

$$x = a \cos \psi \tag{6}$$

$$y = b \sin \psi \tag{7}$$

Differentiating (5), and substituting the values of x and y from (6) and (7)

$$\frac{dy}{dx} = -\frac{b}{a} \cot \psi \tag{8}$$

As CF is normal to the ellipse, it is evident that at the point C

$$\frac{dy}{dx} = \tan(90° - \alpha) = \cot \alpha. \tag{9}$$

Equating the right-hand members of (8) and (9)

$$\cot \psi = a \cot \alpha \div b \tag{10}$$

This gives ψ for the point C, and the corresponding values of x and y, which will be designated x_0 and y_0, are found from (6) and (7). Let D, fig. 7, be any other point in the curve. To find its co-ordinates, produce GC to K, draw DL parallel to AH, and let KJ $= d$. Then

$$\mathrm{ED} = \mathrm{GC} + \mathrm{CK} - \mathrm{KL}. \tag{11}$$

But $\mathrm{ED} = y$, $\mathrm{GC} = y_\circ$, $\mathrm{CK} = d \sec \alpha$, $-\mathrm{KL} = (x - x_\circ) \sin \alpha$, and therefore

$$y = y_\circ + d \sec \alpha + (x - x_\circ) \sin \alpha \tag{12}$$

When y is known (7), gives

$$\sin \psi = y \div b \tag{13}$$

and then x follows from (6). As y can not be accurately found until x is known, we begin by assuming

$$y = y_\circ + d \sec \alpha \tag{14}$$

and having computed the corresponding value of x, a corrected value of y is obtained from (12). This, in its turn, gives a corrected value of x; and by two or three successive approximations, both y and x become known with the utmost exactness.

Putting $\mathrm{MDL} = \beta$, it is evident that because MN is a normal to the ellipse

$$\frac{dy}{dx} = \cot \beta \tag{15}$$

and then, by (8)

$$\cot \beta = - b \cot \psi \div a \tag{16}$$

in which β must be regarded as having the same sign as the y, from which it is derived through (13).

The angle of incidence of the ray KD upon the cornea is MDK, which is equal to $\mathrm{MDL} - \mathrm{KDL}$. But $\mathrm{KDL} = \mathrm{CQA} = \alpha$, and putting $\mathrm{MDK} = \iota$, we have

$$\iota = \beta - \alpha \tag{17}$$

After refraction at the cornea, let the path of the ray KD be DF. Then FDN is the angle of refraction, which we will designate as ι', and if μ is the refractive index of the eye then, by a well-known optical law,

$$\sin \iota' = \sin \iota \div \mu \tag{18}$$

Designating the angle CDE by γ, and the distance CD by D, we have

$$\tan \gamma = \frac{x - x_0}{y - y_0} \qquad (19)$$

$$D = \frac{y - y_0}{\cos \gamma} \qquad (20)$$

From the figure

$$\angle \mathrm{CDF} = \mathrm{CDE} + \mathrm{EDN} + \mathrm{NDF} \qquad (21)$$

But $\mathrm{CDE} = -\gamma$, $\mathrm{EDN} = 90^\circ \sim \beta$, $\mathrm{NDF} = \iota'$, and therefore

$$\angle \mathrm{CDF} = (90^\circ \sim \beta) + \iota' - \gamma \qquad (22)$$

The expression $(90^\circ \sim \beta)$ is used to indicate the excess of 90° over the numerical value of β; the latter quantity being taken without regard to sign. For that portion of the cornea which lies between A and C, the complement of the angle CDF must be employed instead of the angle itself.

Again, from the figure

$$\angle \mathrm{CFD} = \mathrm{MDK} - \mathrm{FDN} = \iota - \iota' \qquad (23)$$

Finally, if F is the point where the ray KD intersects the visual axis, then CF will be the focal distance of the cornea for parallel rays. Representing this distance by F, we have from the triangle CDF

$$F = \frac{y - y_0}{\cos \gamma} \times \frac{\sin [(90^\circ \sim \beta) + \iota' - \gamma]}{\sin [\iota - \iota']} \qquad (24)$$

It yet remains to translate these algebraic formulæ into numbers. We have already obtained the values of a, b, and α for a normal eye, and by substituting them in (10), (7), and (6), we find, for the point where the visual axis enters the cornea,

$$y_0 = 0.78071^{\mathrm{mm.}} \quad x_0 = 10.9227^{\mathrm{mm.}}$$

The monochromatic aberration of the cornea is most

conveniently investigated by tracing the paths which a considerable number of parallel rays impinging upon it will pursue within the eye. Eleven such rays have been considered, all situated in the same horizontal plane, at intervals of half a millimetre from each other, and the central one coinciding with the visual axis. For their passage a pupil five millimetres in diameter is necessary. The eleven rays in question furnish eleven values of d, varying by intervals of half a millimetre from $+2.5^{\text{mm.}}$ to $-2.5^{\text{mm.}}$, from which, by means of formulæ (12), (13), (6), (16), (17), (18), (19), and (24), the corresponding values of F have been computed. With respect to (18), it should be remarked that for reasons already stated, the cornea, the aqueous humor, and the vitreous humor have been regarded as a homogeneous mass for which $\mu = 1.3366$. This is Sir David Brewster's value for the aqueous humor. The value of F for the ray situated in the visual axis has been obtained from the well-known expression.

$$F = \frac{\mu r}{\mu - 1} \tag{25}$$

TABLE I.

KJ.	DE.	HE.	∠ MDL.	∠ MDK.	∠ NDF.
mm.	mm.	mm.			
+2.5	+3.2265	10.2617	+24° 11′ 32″	+18° 21′ 40″	+13° 37′ 54″
2.0	2.7441	.4603	20 32 51	14 42 59	10 57 23
1.5	2.2583	.6250	16 53 36	11 03 44	8 15 13
1.0	1.7[illegible]1	.7566	13 13 28	7 23 36	5 31 26
+0.5	1.2765	.8558	9 32 14	+ 3 42 22	+ 2 46 19
0.0	0.7807	.9227	5 49 52	0 00 00	0 00 00
0.5	+ .2817	.9574	+ 2 06 14	− 3 43 38	− 2 47 16
1.0	− .22[illegible]8	.9593	− 1 38 56	7 28 48	5 35 21
1.5	0.7266	.92[illegible]	5 25 37	11 15 29	8 23 56
2.0	1.2359	.8625	9 14 01	15 03 53	11 12 46
−2.5	−1.7487	10.7614	−13 04 17	−18 54 09	−14 01 37

KJ.	∠ EDC.	Log. CD.	∠ CDF.	∠ CFD.	CF.	CF.
mm.					mm.	mm.
+2.5	−15° 07′ 24″	0.40373	+94° 33′ 46″	4° 43′ 46″	30.630	30.632
2.0	13 15 08	.30473	93 39 40	3 45 36	.697	.697
1.5	11 23 28	.17820	92 45 05	2 48 31	.727	.728
1.0	9 32 22	0.00098	91 50 23	1 52 07	.721	.723
+0.5	− 7 41 05	9.69923	+90 55 10	0 56 03	.682	.684
0.0	. . . .	. . . . .	. . . .	0 00 00	.608	.610
−0.5	− 3 58 41	9.69915	−90 54 49	0 56 22	.504	.502
1.0	2 05 35	0.00094	91 50 50	1 53 27	.357	.359
1.5	− 0 12 05	.17820	92 46 14	2 51 33	30.183	30.182
2.0	+ 1 42 36	.30481	93 41 21	3 51 07	29.970	29.971
−2.5	+ 3 38 56	0.40390	−94 36 16	4 52 32	29.725	29.724

Table I gives for each value of d, the resulting value of F, together with the most important quantities occurring in its computation.

The quantities contained in the various columns are indicated by the letters at their heads, which refer to figure 7. For example, the heading of the first column is KJ, which indicates that the quantities contained in it are the lengths of the line KJ, figure 7. For convenience of reference, the symbols by which these quantities are designated in the formulæ are here recapitulated:

$KJ = d$	Angle $MDL = \beta$
$DE = y$	" $MDK = \iota$
$HE = x$	" $NDF = \iota'$
$CD = (y - y_0) \div \cos\gamma$	" $EDC = \gamma$
$CF = F$	" $CDF = (90 \sim \beta) + \iota' - \gamma$
	" $CFD = \iota - \iota'$

The results contained in the first of the two columns headed CF are exhibited graphically in figure 8, the values of F being taken as abscisses, and the corresponding values of d as ordinates, and it is evident that the focal curve is a

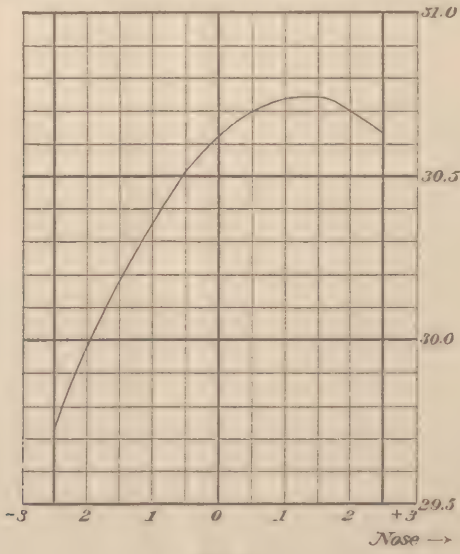

FIG. 8.

parabola. To find an expression for it, we put

$$F = C - m\,(d - n)^2 \qquad (26)$$

developing and rearranging the terms

$$F = (C - mn^2) - md^2 + (2mn)d \qquad (27)$$

When d vanishes (27), reduces to

$$F = (C - mn^2) \qquad (28)$$

and by comparing this with the numbers in the first of the two columns headed CF we see that

$$C - mn^2 = 30.608^{\text{mm.}}$$

Substituting this in (27), and comparing the resulting equation with the values of F for $d = +2.5^{\text{mm.}}$ and $d = -2.5^{\text{mm.}}$, we obtain

$$\begin{aligned} 0 &= -0.022 - 6.25m + 5mn \\ 0 &= +0.883 - 6.25m - 5mn \end{aligned}$$

whence

$$\begin{aligned} m &= +0.0689 & \qquad 2mn &= +0.181 \\ n &= +1.314 & mn^2 &= +0.119 \\ & & C &= +30.727 \end{aligned}$$

Substituting these values in (26) and (27), we have for a first approximation

$$\begin{aligned} F &= 30.727 - 0.0689\,(d - 1.314)^2 \\ &= 30.608 - 0.0689\,d^2 + 0.181\,d \end{aligned}$$

A comparison of the values given by these expressions with those contained in the table, shows that it will be more accurate to take

$$C - mn^2 = 30.610^{\text{mm.}}$$

Doing so, and combining this with the values of F corresponding to $d = +2.0^{\text{mm.}}$, and $d = -2.0^{\text{mm.}}$, we have finally

$$\begin{aligned} F &= 30.730 - 0.0691\,(d - 1.314)^2 \\ &= 30.610 - 0.0691\,d^2 + 0.1817\,d \end{aligned}$$

The values of F computed from these expressions are exhibited in the last column of table I, and the accuracy with which they represent the numbers in the preceding column is apparent.

Table II shows how the monochromatic aberration actually existing in the normal cornea compares with what would have existed if the cornea had been spherical. Upon each line of the table, the first column contains the assumed diameter of the pupil ; the second, third and fourth columns relate to the normal cornea, and contain respectively the greatest and least focal distance occurring within the area of the pupil, and the amount of astigmatism corresponding to them ; the fifth, sixth, and seventh columns relate to a spherical cornea, and contain similar data for it. In computing the focal distance at various points of a spherical cornea it has been assumed that $r = 7.708^{\text{mm.}}$, $\mu = 1.3366$, and the following formulæ have been employed :

$$\sin \iota = d \div r \tag{29}$$

$$\sin \iota' = \sin \iota \div \mu \tag{30}$$

$$F = \frac{2\, r \sin \frac{1}{2}\, \iota \cos \frac{1}{2}\, \iota}{\sin\, (\iota - \iota')} \tag{31}$$

As the monochromatic aberration we are considering occurs in a single meridian, its correction by cylindrical lenses is impracticable, but nevertheless its amount may be expressed in the notation usually employed for astigmatism. The requisite formula is

$$\text{Astigmatism} = 0.0396\ (F - F') \tag{32}$$

In which F and F' are the lengths in millimetres of the greatest and least focal distances found within the area of the pupil, and the result is expressed in terms of the Paris inch.

TABLE II.

d.	Normal Cornea.			Spherical Cornea.		
	F.	*F'.*	As.	*F.*	*F'.*	As.
mm.	mm.	mm.	Par. in.	mm.	mm.	Par. in.
1	30.684	30.502	1:139	30.608	30.556	1:476
2	.723	.359	1: [illegible]	.608	.414	1:13[illegible]
3	.728	30.182	1: 46	.608	30.171	1: 58
4	.728	29.971	1: 33	.608	29.826	1: 32
5	30.728	29.724	1: 25	30.608	29.376	1: 21

In conclusion, the results at which we have arrived may be summed up as follows ;

1. The monochromatic aberration originated by the normal cornea occurs principally on the outer side of the visual axis—that is, on the side farthest from the nose.

2. the diameter of the pupil is usually about four millimetres. For diameters less than this, the monochromatic aberration of a spherical cornea would be less than that of the normal cornea. For greater diameters, the reverse is true. This is contrary to the generally received opinion. (See Donders, foot-note on page 310.)

3. Donders says (pp. 456, 457), the astigmatism in sharp eyes is not generally more than from 1: 140 to 1:60, and whenever it exceeds the latter amount, the power of vision suffers under some circumstances. An astigmatism of 1: 40 he regards as decidedly abnormal. Nevertheless, with a pupil four millimetres in diameter, the normal cornea produces monochromatic aberration to the extent of 1:33 ; and as there is no confusion of images in the normal eye, it seems probable that the crystalline lens exerts some compensating action. This suspicion is strengthened by the well-known fact that in aphakia, the acuteness of vision is nearly always improved by giving a certain inclination to the powerful convex glasses which are then necessary. It therefore become important to ascertain the true structure and position of the lens. What is the distribution of density in its layers? Are its two halves, situated respectively on the inner and outer sides of the visual axis, of unequal refractive power? Is its form unsymmetrical, or is it unsymmetrically placed with respect to the visual axis? Or is it inclined to that axis? These are difficult questions, but doubtless they can be answered by patient observation.[1]

Washington, Feb., 1883.

([1] Prof. Ludwig Matthiessen, of Rostock, has made some extensive investigations on the optical character and properties of the crystalline lens. Compare his exhaustive paper, "Die Differenzialgleichungen der Dioptrik der geschichteten Krystallinse," in E. Pflüger's *Archiv f. Physiologie*, Bd. xix, pp. 480–562 ; also a paper by M. Peschel, "Experimentelle Untersuchungen über die Periscopie der Krystallinse," *ib.*, Bd. xx, pp. 338–353.) S. M. B.

AN HISTORICAL NOTE ON CORNEAL ASTIGMATISM.

Senf was the first (in 1846) to make measurements of the cornea, which showed it to be ellipsoidal rather than spherical in shape. Helmholtz arrived at the same conclusion from his ophthalmometrical measurements which were published in *Gräfe's Archives* B. i, Abt. 2 (1855). In this article (p. 18), he says: "The form of the cornea corresponds approximately to an ellipsoid formed by the revolution of an ellipse about its major axis."

He gives these measurements as well as those of Senf in the first part of his "Physiologische Optik" (pages 8 and 11 of the French edition), published in 1856; but it is evident that he still considered the cornea to be an ellipsoid of revolution, as his measurements were confined to one meridian (the horizontal). He speaks at this time (p. 142) of the astigmatism of Young as being caused by the lens, and of the correction of his own astigmatism (of low degree) by means of an obliquely placed concave lens, but no hint is given that the cause of the astigmatism was in the cornea.

It was Knapp who first determined by ophthalmometric means that the cornea was not an ellipsoid of revolution but an ellipsoid with three unequal axes.

In the "Verhandlungen der vom 3–6 Sept., 1859, im Heidelberg versammelten Augen Artze," Berlin, Peters, 1860, we find (p. 19), that "Dr. Knapp gave an account of his measurements on the curved surface of the human eye, made by means of Helmholtz's ophthalmometer. 1. The cornea. Helmholtz's measurements were confined to the horizontal meridian. Knapp, on the other hand, had measured four eyes in many different meridians with the following result: 1. The centre and apex of the cornea do not coincide. . . . The anterior focal distance of the horizontal ellipse = 23.095 *mm.*; of the vertical meridian = 23.34 *mm.* The posterior focal distance of the horizontal ellipse = 30.18 *mm.*; of the vertical ellipse = 31.1 *mm.* . . . In the discussion on this division of the subject, Knapp remarked that in all probability it was the difference between the vertical and horizontal ellipses which rendered cylindrical glasses necessary,

and was the cause of the difference in the 'accommodation-line' in the vertical and horizontal directions. After cataract-extraction, in sclerectasia and hyperpresbyopia, such glasses were of benefit, as had been shown by Prof. Donders.

"In regard to the accommodation-line, Prof. Donders remarked, that in his opinion, it was due to the lens, from the fact that it was in intimate connection with polyopia, which was undoubtedly caused by the lens as proven by entopic experiments."

These investigations were published in detail by Knapp in his inaugural thesis, "Die Krümmung der Hornhaut des menschlichen Auges" in 1860.

Donders in his first papers on the refraction and accommodation of the eye, published in *Gräfe's Archives* makes in B. vii (1860), Abt. 1, p. 176, an application of this asymmetry to the explanation of abnormal astigmatism, in contradistinction to the lenticular theory of Young, and gives reference to Knapp's paper. So far as I know this is the first mention made by Donders of corneal astigmatism. In *Gräfe's Archives* B. viii (1862), Abt. 2, appeared Knapp's classical paper "Ueber die Assymmetrie des Auges in seinen verscheidenen Meridianebenen." While this paper was in press, Donders published his "Astigmatisme en cylindrische Glazen," which, for the first time, brought the subject of astigmatism and its correction prominently before the profession. Soon afterward (1864), his treatise on the "Anomalies of the Refraction and Accommodation of the Eye," appeared, which made astigmatism a part of the general knowledge of the profession.

The opinion that regular astigmatism resides in the cornea has been most thoroughly substantiated by all observations made since that time. Javal in the *Annales d' oculistique*, t. 87 (1882), pp. 33–43, says that in the testing and measurement of more than 100 eyes, the total astigmatism corresponded exactly with the corneal astigmatism, with the exception of four cases, and in one of these the difference was only 0.2 D; and still further examinations by him, and by Dr. Nordenson, amounting to more than 250 additional cases, have only confirmed his first observations.